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In order that these should be compatible it is necessary and sufficient that

$$\begin{vmatrix} (a + bi) & (c + di) \\ (a' + b'i) & (c' + d'i) \end{vmatrix} = 0.$$

This complex equation is equivalent to the *two* real equations

$$(ac' - a'c) = (bd' - b'd),$$

$$(ad' + bc') = (a'd + b'c).$$

Both of these must be fulfilled if (1) is to subsist.

On the other hand equations (1) are equivalent to

$$\begin{aligned} ap - bq + cr - ds &= 0, \\ bp + aq + dr + cs &= 0, \\ a'p - b'q + c'r - d's &= 0, \\ b'p + a'q + d'r + c's &= 0. \end{aligned}$$

These are compatible when, and only when, a *single* equation is satisfied, namely

$$\begin{vmatrix} a & -b & c & -d \\ b & a & d & c \\ a' & -b' & c' & -d' \\ b' & a' & d' & c' \end{vmatrix} = 0.$$

Which answer is right?

ON A PURELY PROJECTIVE BASIS FOR THE THEORY OF INVOLUTION.

By D. N. LEHMER.

The theory of involution is usually based on foundations which are more or less metrical. It is customary to derive the more important properties by means of circles, or else by the use of the theory of the anharmonic ratio, which itself is usually based on metrical notions of areas and trigonometric functions. These methods have their advantages, but to the purist it seems unfortunate that the general, projective properties of involution should not be derived in a non-metrical way, and that the metrical properties should not be obtained, as is usual, by the introduction of the elements at infinity.

A purely projective discussion of the subject may be based on the following well-known and easily derived theorem on the complete quadrangle:[1]

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[1] See Reye, *Geometry of Position*, Holgate's translation, page 36.

If two complete quadrangles, $KLMN$ and $K'L'M'N'$, are so related that the homologous sides KL and $K'L'$ meet in a point A and the opposite pair, MN and $M'N'$, in A'; LM and $L'M'$ in B, and KN and $K'N'$ in B'; LN and $L'N'$ in C, while A, A', B, B' and C all lie on the same straight line u, then the sixth pair of sides, KM and $K'M'$, will also meet on that line.

The proof follows easily from Desargues's theorem and its converse. For consider the two triangles KLN and $K'L'N'$ (Fig. 1): corresponding sides meet on u; therefore the lines $K'K$, $L'L$, and $N'N$ all meet in a point S. But by the triangles LMN and $L'M'N'$ we observe that MM' must also pass through S. Then since the lines joining corresponding vertices of the triangles KMN and $K'M'N'$ are concurrent in S, we see that the pairs of corresponding sides meet in three collinear points A', B', C' which must all lie on u.

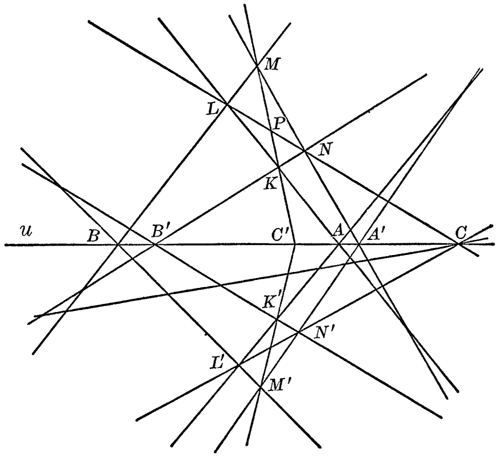

Fig. 1.

In this cut A, A', and B, B' are points on line u and $KLMN$ and $K'L'M'N'$ are complete quadrangles constructed so as to have pairs of opposite sides passing through these points.

From the theorem it is seen that if AA', BB' and C are given then C' is uniquely determined, and that independently of the particular quadrangle employed to construct it. It is clear also that three pairs of points are not generally so placed on a line that the three pairs of opposite sides of a complete quadrangle may be drawn through them. As a definition, then:

If three pairs of points on a line are such that the three pairs of opposite sides of a complete quadrangle may pass through them, the points are said to be three pairs in involution. Any two pairs being given, as many other pairs may be constructed as desired, for choosing any point C, the other point that pairs with it, called its conjugate, may be found by the above linear construction. The solution, in detail, is as follows: Through A and A' draw any two lines. Cut them by any line through C in the points L and N. Join L to B, and N to B', thus obtaining the points M and K on the lines through A' and A respectively. The line MK

determines C'. All the pairs CC' thus obtained are said to belong to the same involution and taken together are said to separate the line in involution.

As C traverses the line, C' also traverses the line and the two point-rows thus obtained are projective. For fix K and N, then CN cuts out on KA a point-row L, which projects from B to a point-row M on NA', which, in turn, projects from K to C'.

It follows from this last remark that NC and KC' furnish two projective pencils, and that their intersection, P, generates a conic. This conic meets the line u in two points F_1 and F_2, real or imaginary, which are the *double points* in the involution; that is, they coincide with their conjugate points. Therefore: *An involution has two double points, or none.*

B and B' being a pair in the involution, the rays NB and KB' are corresponding rays. Therefore NB is tangent to the conic at N, and KB is tangent to it at K. NK is then the polar of B with respect to the conic. B and B' are thus conjugate with respect to the conic, and are thus harmonically separated by the two double points. We have then the theorem:

Conjugate points in an involution are harmonic conjugates with respect to the double points.

It is to be noted that, if the complete quadrangle is drawn with two pairs of opposite sides passing through the double points, the figure reduces to that used in defining four harmonic points.

Metrical theory. We introduce now that point at infinity the conjugate of which we will call the *center* and designate by O. Since conjugate points are harmonically separated by the double points it follows that O is the middle point of F_1F_2.

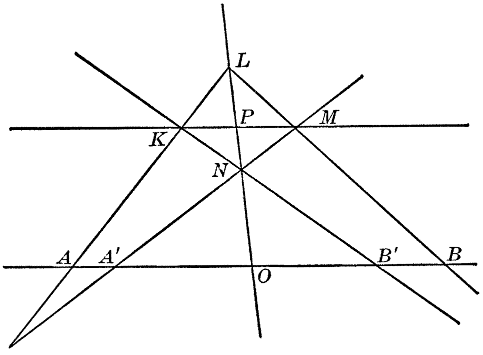

Fig. 2.

In constructing the center, the line MK (Fig. 2) is drawn parallel to u. It is then seen that the triangle MPL is similar to the triangle OLB, and the triangle PNK to the triangle ONB'; whence we may write

$$\frac{OB}{MP} = \frac{OL}{PL}$$

and

$$\frac{OB'}{PK} = \frac{ON}{PN};$$

whence

$$\frac{OB \cdot OB'}{MP \cdot PK} = \frac{OL \cdot ON}{PL \cdot PN}. \tag{1}$$

Also from the similar triangles OAL and PKL, combined with the similar triangles ONA' and MPN, we have

$$\frac{OA \cdot OA'}{MP \cdot PK} = \frac{OL \cdot ON}{PL \cdot PN}. \tag{2}$$

By combining (1) and (2) we have the fundamental relation

$$OA \cdot OA' = OB \cdot OB',$$

or in words:

The product of the distances from the center to a pair of conjugate points in the involution is constant.

From this theorem all of the usual developments may easily be made. The corresponding theory of lines in involution may be obtained in the same way, or by applying the principal of duality.

BOOK REVIEWS.

Edited by W. H. Bussey.

The Pell Equation. By Edward Everett Whitford, Instructor in Mathematics in the College of the City of New York. Whitford, New York, 1912. iv + 193 pages. $1.00 postpaid.

Let A be a given non-square positive integer and consider the equation

$$x^2 - Ay^2 = 1 \tag{1}$$

from the point of view of determining the positive integers x, y which satisfy the equation. This is the so-called Pellian problem. Equation (1) is referred to as the Pell equation.

The principal purpose of the book under review is to give a history of the Pell equation (1) and of the more general equation $x^2 - Ay^2 = B$, where B is a given integer (positive or negative). There is added a table of the simplest solution of (1) for each value of A from 1,501 to 1,700 inclusive, similar tables having previously been given by other authors for the values of A less than 1,501. The book contains also a bibliography of the Pell equation, with references to over 300 authors, and a table of continued fractions for $\sqrt{A}$.